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Subject: Physics

Production of Courseware

e-Content for Post Graduate Courses

Paper No. : Classical Mechanics

Module : Symmetry Transformations and Conservation Laws

Development Team

Principal Investigator	Prof. Vinay Gupta, Department of Physics and Astrophysics, University of Delhi, Delhi
Paper Coordinator	Ashok Goyal, Department of Physics and Astrophysics, University of Delhi, Delhi
Content Writer	Ashok Goyal, Department of Physics and Astrophysics, University of Delhi, Delhi
Content Reviewer	

Description of Module	
Subject Name	Physics
Paper Name	Classical Mechanics
Module Name/Title	Symmetry transformations and Conservation Laws
Module Id	3

Symmetry transformations and conservation Laws

Content:

1. Introduction
2. Space-time translational symmetry
3. Rotational symmetry
4. Noether's Theorem
5. Summary

Learning objective

- Learn about the deep relationship between symmetry and conservation laws.
- Learn about Noether's theorem which says that the invariance of Lagrangian under continuous. Symmetry operation leads to a corresponding conserved quantity.

1. Introduction:

Principle of symmetry transformations plays a key role in the formulation of Quantum Field theories in many branches of physics like particle physics, condensed matter physics, general theory of relativity, string theories etc. Conservation laws in physics have a deep relationship with symmetry operations. The requirement of invariance of the Lagrangian under symmetry transformations so that the physical results remain unchanged leads to conservation laws. In what fallows we will consider some space time transformations leading to corresponding conservation laws. Suppose we have a Lagrangian of the system $L(q, \dot{q}, t)$ where q's stand for the set generalized coordinates. If we now make a transformation of the coordinate from $q \to q'$, the Lagrangian in new coordinates is expressed as $L'(q', \dot{q}', t)$. The different solutions $q(t)$ obtained from L and L' must have equivalent trajectories (path). The coordinates transformations can be discrete or continuous. An example of a discrete transformation is 'mirror reflection' under which

$$x \to x' = -x, \quad y \to y' = -y \quad \text{and } z \to z' = z \tag{3.1}$$

or a translation under which, for example

$$x \to x' = x + a, \quad y \to y' = y, \quad z \to z' = z \tag{3.2}$$

A transformation is continuous if the transformed coordinates have a continuous dependence on a suitable parameter defining the transformation. An example is rotational transformation about the z-axis under which

$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

$$z' = z \tag{3.3}$$

2. Space time Translational symmetry and conservation of Energy and Linear Momentum.

The space is assumed to be homogeneous and isotropic. A physical system at one place in space and time developes the same way as at any other point. The laws of physics are invariant under the space-time translation. Let us consider a Lagrangian of a dynamical system of n particles. The Lagrangian is a function of position coordinates and velocities. Assuming the constraints to be holonomic, the Lagrangian can be written as

$$L(\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i) = T - U(\boldsymbol{r}_i) \tag{3.4}$$

Where the potential does not depend on the velocities.

1) **Symmetry under space translation:**

Consider an infinitesimal translation of the coordinate

$$\boldsymbol{r} \to \boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{a} \tag{3.5}$$

Where $\boldsymbol{a}$ is a constant vector. Under this translation $L \to L + \delta L$ where

$$\delta L = \sum_i \frac{\partial L}{\partial \boldsymbol{r}_i} \cdot \boldsymbol{a} \qquad (3.6)$$

If the Lagrangian remains invariant so that the physical results remain unchanged, we have $\delta L = 0$. Since $\boldsymbol{a}$ is an arbitrary displacement

$$\sum_i \frac{\partial L}{\partial \boldsymbol{r}_i} = 0 \qquad (3.7)$$

and from Lagrange's equation of motion

$$\frac{d}{dt}\sum_i \frac{\partial L}{\partial \dot{\boldsymbol{r}}_i} = 0 \qquad (3.8)$$

Now

$$L = \sum_i \frac{1}{2} w_i \dot{\boldsymbol{r}}_i^2 - U(\boldsymbol{r}_i)$$

$$\frac{\partial L}{\partial \dot{\boldsymbol{r}}_i} = \sum_i m_i \cdot \dot{\boldsymbol{r}}_i = \sum_i \dot{\boldsymbol{p}}_i \qquad (3.9)$$

From (3.8)

$$\frac{d}{dt}\sum_i \boldsymbol{p}_i = 0 \quad \text{i. e.} \quad \sum_i \boldsymbol{p}_i = \text{constant} \qquad (3.10)$$

Total linear momentum of the system is conserved. IF the motion of the system is described by a set of generalized coordinates $\{q_k\}$ we define the quantity

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad (3.11)$$

as 'generalized momenta' or 'conjugate momenta. In general the conjugate momenta may not be identical to the mechanical momenta and may have dimensions different from MLT^{-1} . The dimension of mechanical momenta. In terms of generalized force $f_i = \frac{\partial U}{\partial q_i}$ the Lagrangian equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}_i}\right) = \dot{p}_i = f_i \qquad (3.12)$$

If in the expression of the Lagrangian, a particular coordinate q_l does not appear explicitly, it is called 'cyclic' or 'ignorable' and obviously for a cyclic coordinate

$$\frac{\partial U}{\partial q_l} = \frac{\partial L}{\partial q_l} = 0 \quad \text{and} \quad \frac{d}{dt}(p_i) = 0 \tag{3.13}$$

Hence the momentum conjugate to acyclic coordinate is conserved.

ii) Symmetry under time translation.

Under time translation

$$t \to t' = t + \delta t \tag{3.14}$$

The Lagrangian is transformed to $L \to L + \delta L$

where

$$\delta L = \frac{\partial L}{\partial t}\delta t \tag{3.15}$$

If the Lagrangian remains invariant under time translation $\partial L = 0 = \frac{\partial L}{\partial t}$

Now

$$\frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial \boldsymbol{r}_i}\frac{\partial \boldsymbol{r}_i}{\partial t} + \frac{\partial L}{\partial \dot{\boldsymbol{r}}_i}\frac{\partial \dot{\boldsymbol{r}}_i}{\partial t}\right) + \frac{\partial L}{\partial t}$$

$$= \sum_i \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}_i}\right)\frac{\partial \boldsymbol{r}_i}{\partial t} + \frac{\partial L}{\partial \dot{\boldsymbol{r}}_i}\frac{\partial \dot{\boldsymbol{r}}_i}{\partial t}\right) + \frac{\partial L}{\partial t}$$

Where we have used the Lagrange equation in the first term on the right to express $\frac{\partial L}{\partial \boldsymbol{r}_i}$ as $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}_i}\right)$

$$\therefore \frac{dL}{dt} = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}_i}\dot{\boldsymbol{r}}_i\right) + \frac{\partial L}{\partial t}$$

$$\frac{d}{dt}\left\{L - \sum_i \frac{\partial L}{\partial \dot{\boldsymbol{r}}_i}\dot{\boldsymbol{r}}_i\right\} = -\frac{\partial L}{\partial t} = 0 \tag{3.16}$$

Now

$$L = T - U(\boldsymbol{r}_i) = \frac{1}{2}\sum_i m_i \dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i - U(\dot{\boldsymbol{r}}_i)$$

$$\frac{\partial L}{\partial \dot{\boldsymbol{r}}_i} = \sum_i m_i \dot{\boldsymbol{r}}_i$$

U does not depend on $\dot{\boldsymbol{r}}_i$.

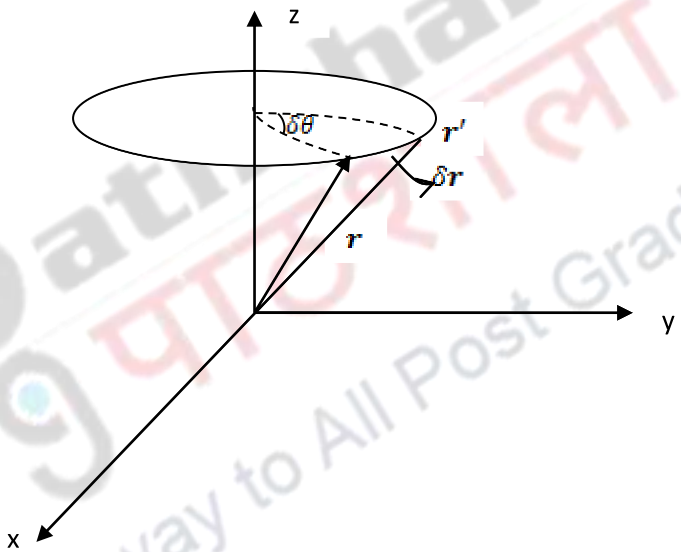
$$\therefore \frac{d}{dt}\left(L - \sum_i m_i \dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i\right) = \frac{d}{dt}\left(T - U - \sum_i m_i \dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i\right) = 0$$

$$\frac{d}{dt}(T - U - 2T) = 0 \Longrightarrow \frac{d}{dt}(T + U) = 0 \qquad (3.17)$$

which implies that the total energy is conserved.

3. Rotational symmetry

Let a vector $\boldsymbol{r}$ rotates around the z-axis and rotates to $\boldsymbol{r}'$ after rotating by an angle δ as shown. The tip of the vector $\boldsymbol{r}$ moves along a circle of radius $\boldsymbol{r}\sin\theta$.



$$\boldsymbol{r} \to \boldsymbol{r}' = \boldsymbol{r} + \delta\boldsymbol{r} \qquad (3.18)$$

$$\delta\boldsymbol{r} = \delta\boldsymbol{\theta} \times \boldsymbol{r} \qquad (3.19)$$

Langrangi equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}_i}\right) - \frac{\partial L}{\partial \boldsymbol{r}_i} = 0$$

Under rotation $L \to L' + \delta L$ where

$$\delta L = \sum_i \left(\frac{\partial L}{\partial \boldsymbol{r}_i}\partial \boldsymbol{r}_i + \frac{\partial L}{\partial \dot{\boldsymbol{r}}_i}\partial \dot{\boldsymbol{r}}_i\right)$$

$$= \sum_i \left\{ \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{r}}_i}(\delta\boldsymbol{\theta} \times \boldsymbol{r}) - \frac{\partial L}{\partial \dot{\boldsymbol{r}}_i}\cdot(\delta\boldsymbol{\theta} \times \dot{\boldsymbol{r}}_i) \right\} \qquad (3.20)$$

Since

$$L = \sum_i \left\{ \frac{1}{2} m_i \partial\dot{\boldsymbol{r}}_i \cdot \partial\dot{\boldsymbol{r}}_i - U(\boldsymbol{r}_i) \right\}$$

$$\frac{\partial L}{\partial \dot{\boldsymbol{r}}_i} = \boldsymbol{p}_i \qquad (3.21)$$

$$\therefore \delta L = \sum_i \dot{\boldsymbol{p}}_i \cdot (\delta\boldsymbol{\theta} \times \boldsymbol{r}) + \dot{\boldsymbol{p}}_i \cdot (\delta\boldsymbol{\theta} \times \dot{\boldsymbol{r}}_i)$$

$$= \sum_i \delta\boldsymbol{\theta} \cdot (\boldsymbol{r} \times \dot{\boldsymbol{p}}_i + \dot{\boldsymbol{r}}_i \boldsymbol{p}_i) = \sum_i \delta\boldsymbol{\theta} \cdot \frac{d}{dt}(\boldsymbol{r}_i \times \boldsymbol{p}_i) \qquad (3.22)$$

If the Lagrangian is invariant under rotation

$$\sum_i \delta\boldsymbol{\theta} \cdot \frac{d}{dt}(\boldsymbol{r}_i \times \boldsymbol{p}_i) = 0 \qquad (3.23)$$

But $\boldsymbol{r}_i \times \boldsymbol{p}_i = \boldsymbol{l}_i$ the angular momentum of the i[th] particle we get

$$\frac{d}{dt}\sum_i \boldsymbol{l}_i = 0 \qquad (3.24)$$

Total angular momentum is conserved.

Invariance of the Lagrangian under rotation is justified because of the isotropy of space.

4. Neother's Theorem

Suppose we have a Lagrangian which depends on the generalized coordinates, generalized velocities and time i.e. $L(\{q_k\}\{\dot{q}_k\},t)$ and a continuous symmetry operation denoted by $S(\{\alpha_k\},t)$ which depends on the parameter set $\{\alpha_k\}$ and t. For example, the symmetry operation of rotation about the Z-axis is parameterized by a continuous angle of rotation θ. The transformed Lagrangian L' is a function of $S(\alpha,t)$ and $\dot{S}(\alpha,t)$ S i.e.

$$L' = L\big(S(\alpha,t), \dot{S}(\alpha,t), t\big) = 0 \qquad (3.25)$$

If $S(\alpha,t)$ is a symmetry operation of the system, we must have

$$\frac{d}{dS} = L\big(S(\alpha,t), \dot{S}(\alpha,t), t\big) = 0 \qquad (3.26)$$

Thus we must have

$$\frac{\partial L}{\partial S}\frac{\partial S}{\partial \alpha}+\frac{\partial L}{\partial \dot{S}}\frac{\partial \dot{S}}{\partial \alpha}=0 \qquad (3.27)$$

Using Euler-Lagrange equation

$$\frac{\partial L}{\partial S}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{S}}\right)=0 \qquad (3.28)$$

To replace $\frac{\partial L}{\partial s}$ in (3.27) we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{S}}\right)\frac{\partial S}{\partial \alpha}+\frac{\partial L}{\partial \dot{S}}\frac{\partial \dot{S}}{\partial \alpha}=0$$

i.e.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{S}}\ \frac{\partial S}{\partial \alpha}\right)=0 \qquad (3.29)$$

We thus have

$$\frac{\partial L}{\partial \dot{S}}\ \frac{\partial S}{\partial \alpha}=\text{constant} \qquad (3.30)$$

We can define a quantity

$$I_j(\{q_k\},\{\dot{q}_k\},t)\equiv\sum_k \frac{\partial L}{\partial \dot{S}_k}\frac{\partial S_k}{\partial \alpha_j}\bigg|_{\alpha_j=0}=\text{constant} \qquad (3.31)$$

Since eqn. (3.30) holds for all α's.

This is **Noether's Theorem**.

Noether's Theorem states that if a Lagrangian is invariant under a continuous symmetry operation parameterized by M parameters $\alpha_1, \alpha_2, \dots\dots\dots\dots \alpha_M$, then there are M conserved quantities associated with the symmetry transformation.

This theorem formed the corner stone in the development of Quantum Field Theory.

Example: Consider a system with the Lagrangian

$$L=\frac{1}{2}\sum m_i\left(\dot{r}_i^2+\dot{r}_i^2\dot{\theta}_i^2+\dot{z}_i^2\right)-U(r)$$

(r, θ, z) are the generalized coordinates and the potential depends only on T.

The Lagrangian does not involve the coordinates θ and z, therefore they are cyclic coordinates. We then have the momenta conjugate to θ and z. Conserved.

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \sum_i m_i r_i^2 \dot{\theta}_i = l_z$$

the angular momentum of the system about the z-axis, since l_z is defined as

$$l_z = \sum_i m_i (\boldsymbol{r}_i \times \boldsymbol{v}_i)_z = \sum m_i\, (\boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i)_z$$

$$= \sum_i m_i (x_i \cdot \dot{y}_i - y_i \dot{x}_i)$$

Expressing in terms of generalized coordinates r, θ and z $x = r\cos\theta,\ \ y = r\sin\theta,\ z = z$

$$l_z = \sum_i m_i r_i^2 \dot{\theta}_i$$

The momenta conjugate to z is

$$p_z = \frac{\partial L}{\partial \dot{z}} = \sum_i m_i \dot{z}_i$$

which is nothing but the linear momentum of the system along the z-axis. Thus the linear and angular momentum of the system about the z-axis are conserved.

5. Summary:

Homogeneity and Isotropy of space leads to the conservation of energy linear momentum and angular momentum shown in the table.

Symmetry	Lagrangian	Conserved quantity
1. Space translation Homogeneity	Invariant under space transition	Linear momentum

2. Time translation homogeneity of time	Invariant under translation in time and no explicit dependence on time	Total Energy
3. Rotation Isotropy of space	Invariant under rotation	Angular momentum

There are seven constraints (integrals of motion) for a closed system; total energy, three components of linear momentum and three components of angular momentum.

- If a Lagrangian is invariant under a continuous transformation parameterized by M parameters, then according to Noether's theorem, there are M conserved quantities associated with these symmetry transformations.